

Particle scattering by a test fluid on a Schwarzschild spacetime: the equation of state matters

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Abstract The motion of a massive test particle in a Schwarzschild spacetime surrounded by a perfect fluid with equation of state $p_0 = w\rho_0$ is investigated. Deviations from geodesic motion are analyzed as a function of the parameter w , ranging from $w = 1$ which corresponds to the case of massive free scalar fields, down into the so-called “phantom” energy, with $w < -1$. It is found that the interaction with the fluid distribution leads to capture (escape) of the particle trajectory in the case $1 + w > 0$ (< 0), respectively. Based on this result, it is argued that inspection of the trajectories of test particles in the vicinity of a Schwarzschild black hole may offer a new means of gaining insights into the nature of cosmic matter.

Keywords Scattering of particles · Schwarzschild spacetime · Poynting-Robertson-like effects

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1 Introduction

Material bodies moving within a surrounding fluid are typically subject to a force associated with the fluid-body interaction. For instance, in the case of a photon gas this force acts like a friction contrasting the particle motion, as first considered by Poynting [1] and Robertson [2]. The force term entering the equations of motion was given there by the 4-momentum density of radiation observed in the particle’s rest frame with a multiplicative constant factor expressing the strength of the interaction itself. In the present paper we follow the same approach to investigate the general features

of particle motion in a Schwarzschild spacetime region filled with a fluid with equation of state $p_0 = w\rho_0$. The parameter w characterizes different kind of fluids, from massive free scalar fields ($w = 1$), to non-relativistic dust ($w = 0$), all the way down deep into the so-called “phantom” energy ($w < -1$). As it is well known from mathematical cosmology [3], the parameter w has a profound influence on the time evolution of the Universe. For instance, in the Friedmann-Robertson-Walker (FRW) models it determines the power law exponent of the scale factor as a function of time. According to our analysis, which relies on an explicit form of the fluid-particle interaction, the parameter w plays a major role also in discriminating among different types (escape versus capture) of particle motion, for a given black hole spacetime metric. In particular, it is shown that the condition $w + 1 = 0$ (the case of a cosmological constant) identifies with geodesic motion in the Schwarzschild metric, and sets the borderline between capture (positive sign) and escape (negative sign), respectively. This sharp borderline could in principle lead to measurable effects, hence new hints on the nature of the cosmological matter, based on the observation of the motion of test particles embedded in the cosmological fluid surrounding a Schwarzschild black hole.

2 Test fluid superposed to a Schwarzschild spacetime

We begin by writing the Schwarzschild metric in the standard form

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

with the “lapse function” N given by

$$N = \sqrt{1 - \frac{2M}{r}}, \quad (2)$$

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where M is the mass of the black hole. The metric (1) is static, i.e., $\xi^\mu = (\partial_t)^\mu$ is a hypersurface-forming time-like Killing vector. Observers at rest with respect to the coordinates (or “static observers”) have their 4-velocity vector aligned along the Killing direction itself, namely

$$u = N^{-1} \partial_t. \quad (3)$$

An orthonormal frame $e_{\hat{a}}$ ($a = 1, 2, 3$) adapted to them is given by

$$e_{\hat{r}} = N \partial_r, \quad e_{\hat{\theta}} = \frac{1}{r} \partial_\theta, \quad e_{\hat{\phi}} = \frac{1}{r \sin \theta} \partial_\phi. \quad (4)$$

For later use, let us introduce co-rotating (+) and counter-rotating (−) circular timelike geodesic orbits on the equatorial plane $\theta = \pi/2$, characterized by the “Keplerian” 4-velocity

$$U_K = \gamma_K (u \pm \nu_K e_{\hat{\phi}}) = \Gamma_K (\partial_t \pm \zeta_K \partial_\phi), \quad (5)$$

where $\gamma_K = (1 - \nu_K^2)^{-1/2}$ and $\Gamma_K = \gamma_K / N$, with

$$\nu_K = \sqrt{\frac{M}{r - 2M}}, \quad \gamma_K = \sqrt{\frac{r - 2M}{r - 3M}}. \quad (6)$$

These orbits only exist for $r > 3M$. Next, let us consider a (test) perfect fluid, superposed to the Schwarzschild spacetime, described by the following stress-energy tensor [4]

$$T^{\mu\nu}(r) = \rho_0(r) [(1 + w) u^\mu u^\nu + w g^{\mu\nu}], \quad (7)$$

i.e., with equation of state $p_0(r) = w \rho_0(r)$, where w is a constant characterizing the type of cosmic fluid, as detailed below. The form of $\rho_0(r)$ is obtained by integrating the equations $\nabla_\nu T^{\mu\nu} = 0$; in this case, due to spherical symmetry of the background, the solution of these equations is easily obtained, i.e.,

$$\rho_0(r) = \rho_0^\infty N^{-\frac{(w+1)}{w}}, \quad (8)$$

where ρ_0^∞ is a constant representing the value of the energy density at infinity. Noted values of w in the literature (see, e.g., Refs. [5, 6, 7]) are as follows:

- $w = 1$: massive free scalar field;
- $w = 1/3$: radiation;
- $w = 0$: dust (non-relativistic matter);
- $-1/3 < w < -1$: quintessence;
- $w = -1$: cosmological constant;
- $w < -1$: phantom energy.

As is well known, each of these values corresponds to a different growth of the cosmological scale factor in the simple FRW model, that is $R(t) = R(t_0)(t/t_0)^{\frac{2/3}{1+w}}$. This shows that the capture scenario, i.e., $1 + w > 0$, corresponds to initial time singularities (Big-Bang), whereas the escape scenario, i.e., $1 + w < 0$, associates with finite-time future singularities (Big-Rip) (see Ref. [8] and references therein).

3 Scattering of particles

Based on the energy momentum tensor (7), we model the force acting on a massive particle with 4-velocity $U^\alpha = dx^\alpha/d\tau$, in the form of a linear drag term, proportional to the particle velocity

$$f_{(\text{scat})}(U)_\alpha = -\sigma P(U)_{\alpha\beta} T^\beta{}_\mu U^\mu, \quad (9)$$

where

$$U = \gamma(u + \nu^{\hat{a}} e_{\hat{a}}), \quad \gamma = (1 - \delta_{\hat{a}\hat{b}} \nu^{\hat{a}} \nu^{\hat{b}})^{-1/2}, \quad (10)$$

$P(U) = g + U \otimes U$ projects orthogonally to U and σ is a numerical coefficient. This expression for the scattering force has been already used in the literature to study radiation scattering [1, 2, 9, 10, 11, 12]. In that case, σ models the absorption and consequent re-emission of radiation by the particle. Here, we extend its form in such a way as to cover more general situations. The entire discussion about deviations from geodesic motion given below relies on this choice of the force. Although by no means unique, the definition (9) has the merit of mathematical simplicity, as combined with a clear physical basis.

The motion of a massive particle is thus governed by the equations

$$ma(U) = f_{(\text{scat})}(U), \quad (11)$$

where $a(U) = \nabla_U U$ denotes the particle 4-acceleration with frame components

$$a(U)^{\hat{t}} = \frac{d\gamma}{d\tau} + \frac{\gamma^2 N}{r} \nu_K^2 \nu^{\hat{r}}, \quad (12)$$

$$a(U)^{\hat{r}} = \frac{d(\gamma \nu^{\hat{r}})}{d\tau} - \frac{\gamma^2 N}{r} (\nu^{\hat{\theta}2} + \nu^{\hat{\phi}2} - \nu_K^2),$$

$$a(U)^{\hat{\theta}} = \frac{d(\gamma \nu^{\hat{\theta}})}{d\tau} + \frac{\gamma^2}{r \sin \theta} [N \sin \theta \nu^{\hat{r}} \nu^{\hat{\theta}} - \cos \theta \nu^{\hat{\phi}2}],$$

$$a(U)^{\hat{\phi}} = \frac{d(\gamma \nu^{\hat{\phi}})}{d\tau} + \frac{\gamma^2}{r \sin \theta} [N \sin \theta \nu^{\hat{r}} + \cos \theta \nu^{\hat{\theta}}] \nu^{\hat{\phi}}.$$

The quantities ν_K and γ_K have been introduced in Eq. (6). From the orthogonality condition $U \cdot a(U) = 0$, the following relation holds

$$a(U)^{\hat{t}} = \nu_{\hat{r}} a(U)^{\hat{r}} + \nu_{\hat{\theta}} a(U)^{\hat{\theta}} + \nu_{\hat{\phi}} a(U)^{\hat{\phi}}. \quad (13)$$

In the sequel, we shall focus on the special case of a particle confined to the equatorial plane ($\theta = \pi/2$), namely with $\nu^{\hat{\theta}} = 0$. In this case, $a(U)^{\hat{\theta}} = 0$ and the remaining components simplify to

$$a(U)^{\hat{t}} = \frac{d\gamma}{d\tau} + \frac{\gamma^2 N}{r} \nu_K^2 \nu^{\hat{r}},$$

$$a(U)^{\hat{r}} = \frac{d(\gamma \nu^{\hat{r}})}{d\tau} - \frac{\gamma^2 N}{r} (\nu^{\hat{\phi}2} - \nu_K^2),$$

$$a(U)^{\hat{\phi}} = \frac{d(\gamma \nu^{\hat{\phi}})}{d\tau} + \frac{\gamma^2 N}{r} \nu^{\hat{r}} \nu^{\hat{\phi}}. \quad (14)$$

The general equatorial motion is thus fully described by the following equations

$$\begin{aligned}\frac{d\nu^{\hat{r}}}{d\tau} &= \frac{1}{\gamma} \left(\frac{a(U)_{\hat{r}}}{\gamma_{\hat{r}}^2} - a(U)_{\hat{\phi}} \nu^{\hat{r}} \nu^{\hat{\phi}} \right) \\ &\quad + \frac{\gamma N}{r} [\nu^{\hat{\phi}2} - \nu_K^2 (1 - \nu^{\hat{r}2})], \\ \frac{d\nu^{\hat{\phi}}}{d\tau} &= \frac{1}{\gamma} \left(\frac{a(U)_{\hat{\phi}}}{\gamma_{\hat{\phi}}^2} - a(U)_{\hat{r}} \nu^{\hat{r}} \nu^{\hat{\phi}} \right) - \frac{\gamma N}{\gamma_K^2 r} \nu^{\hat{r}} \nu^{\hat{\phi}},\end{aligned}\quad (15)$$

where $\gamma_r = 1/\sqrt{1 - \nu^{\hat{r}2}}$ and $\gamma_{\phi} = 1/\sqrt{1 - \nu^{\hat{\phi}2}}$, once the acceleration components are replaced by the force components according to

$$\begin{aligned}a(U)^{\hat{r}} &= \frac{1}{m} f_{(\text{scat})}(U)^{\hat{r}} = -\frac{\sigma}{m} (w+1) \rho_0^{\infty} \gamma^3 \nu^{\hat{r}}, \\ a(U)^{\hat{\phi}} &= \frac{1}{m} f_{(\text{scat})}(U)^{\hat{\phi}} = -\frac{\sigma}{m} (w+1) \rho_0^{\infty} \gamma^3 \nu^{\hat{\phi}}.\end{aligned}\quad (16)$$

The evolution equations for the radial and azimuthal coordinates follow directly from the definition of U (see Eq. (10)), i.e.,

$$\frac{dr}{d\tau} = \gamma N \nu^{\hat{r}}, \quad \frac{d\phi}{d\tau} = \frac{\gamma}{r} \nu^{\hat{\phi}}. \quad (17)$$

We note that when $w = -1$, the force term $f_{(\text{scat})}(U)$ vanishes identically, implying geodesic motion. Therefore, if the superposed fluid corresponds to a cosmological constant term in the Einstein's equations, the interaction between particles and gas cannot be modeled by Eq. (9). This is exactly the case of the de Sitter spacetime. In fact, for $T_{\mu\nu} = -\Lambda g_{\mu\nu}$, Λ being the cosmological constant, we have

$$T_{\mu\nu} U^{\nu} = -\Lambda U_{\mu}, \quad (18)$$

which vanishes upon projecting orthogonally to U . In this case, a more elaborate force term should be used. Let us further note that at $w = -1$ there is a sign change in the force, yielding an outward push as $w < -1$. This sign change holds the key to the transition from capture to escape scenarios.

Moving to a polar representation of the velocity, i.e., $\nu^{\hat{r}} = \nu \sin \alpha$ and $\nu^{\hat{\phi}} = \nu \cos \alpha$, the above equations take the following form

$$\begin{aligned}\frac{d\nu}{d\tau} &= -A(w+1) \nu N - \frac{(w+1)}{w} - \gamma^{-1} \sin \alpha \frac{M}{r^2 N}, \\ \frac{d\alpha}{d\tau} &= \gamma \frac{N}{r \nu} \cos \alpha (\nu^2 - \nu_K^2), \\ \frac{dr}{d\tau} &= \gamma N \nu \sin \alpha, \quad \frac{d\phi}{d\tau} = \frac{\gamma}{r} \nu \cos \alpha,\end{aligned}\quad (19)$$

where the interaction parameter

$$A = \frac{\sigma}{m} \rho_0^{\infty}, \quad (20)$$

has been introduced so that $A = 0$ identifies the equatorial geodesic case. Note that the presence of A affects

only the first equation of (19). Apart from very special situations, the analysis of these equations can only be performed numerically. Figs. 1 and 2 show typical orbits for the same choice of initial conditions, but different values of the interaction strength parameter A , at different values of the parameter w . Simple inspection shows that values of $w > -1$ ($w < -1$) imply capture (escape) from the hole, as already stated. In Fig. 1, escaping orbits are characterized by a monotonically increasing speed, up to the speed of light. This does not occur in Fig. 2, due to the very small value of the coupling parameter A . In both cases, however, the deviations from geodesic motion are clearly appreciated.

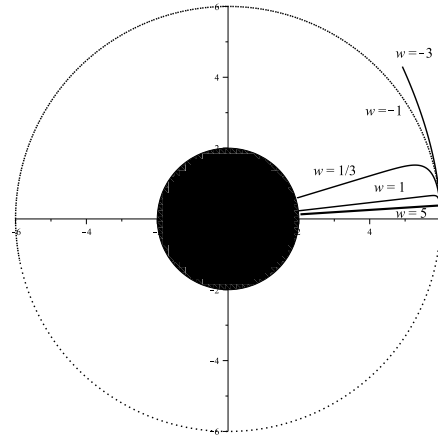


Fig. 1 Motion of a particle undergoing non-geodesic motion due to pressure forces in a Schwarzschild spacetime with a superposed fluid. The black disk denotes the Schwarzschild horizon, located at $r = 2M$. The solid lines starting at $r/M = 6$ (with $\phi_0 = 0$, $\nu_0 = \nu_K$, $\alpha_0 = 0$) represent the particle motion associated with the scattering of the fluid, with parameters $A = 0.1$ and $w = [-3, 1/3, 1, 5]$. The short-dashed curve corresponds to $w = -1$, hence to geodesic motion. With this choice of A , all particles scattered by a fluid with $w+1 > 0$ fall into the hole. Negative values of $w+1$, instead, allow particles to escape capture, and imply a sudden increase of the speed, which reaches the speed of light, i.e., $\nu = 1$, in a finite time (in order to preserve the causality condition for the particle world line, the numerical integration has been stopped at $\nu = 1$).

3.1 Scattering by multi-component matter

In principle, it is interesting to consider multi-component fluid scenarios, in which the matter around the hole corresponds to regions of different fluids (with different w), superposed to the Schwarzschild background. This is only an approximation, since the various (ideal) fluids in the different regions would rapidly mix and

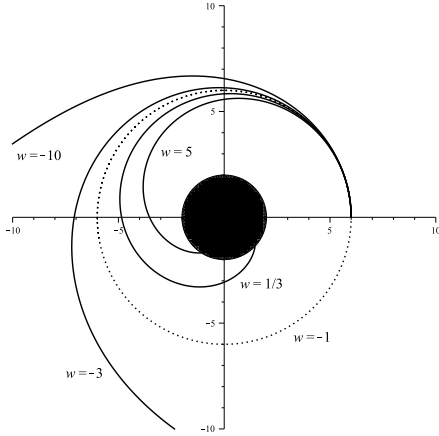


Fig. 2 The same as in Fig. 1, but with $A = 10^{-3}$ and $w = [-10, -3, 1/3, 5]$. With this choice of A (denoting a weaker interaction with respect to the previous case), escaping particles ($w < -1$) never reach the speed of light.

yield a mixture with intermediate values of w . However, if only as a hypothetical speculation, it is of interest to explore the way the equation for ν (and only that) slightly modifies to account for the presence of multi-component matter, say with \mathcal{N} components, associated with the parameters $A_1 \dots A_{\mathcal{N}}$ and $w_1 \dots w_{\mathcal{N}}$. The new evolution equation for ν is thus given by

$$\frac{d\nu}{d\tau} = - \sum_{i=1}^{\mathcal{N}} A_i (w_i + 1) \nu N^{-\frac{(w_i+1)}{w_i}} - \gamma^{-1} \sin \alpha \frac{M}{r^2 N}. \quad (21)$$

Explicitly, for $\mathcal{N} = 2$ we have, for example,

$$\begin{aligned} \frac{d\nu}{d\tau} = & -A_1 (w_1 + 1) \nu N^{-\frac{(w_1+1)}{w_1}} - A_2 (w_2 + 1) \nu N^{-\frac{(w_2+1)}{w_2}} \\ & - \gamma^{-1} \sin \alpha \frac{M}{r^2 N}. \end{aligned} \quad (22)$$

This is equivalent to a heterogeneous mixture with a radially-dependent w parameter, namely

$$w(r) = \frac{A_1 N_1(r) w_1 + A_2 N_2(r) w_2}{A_1 N_1(r) + A_2 N_2(r)}, \quad (23)$$

where we have set $N_k(r) \equiv N(r)^{-(1+w_k)/w_k}$. One may then consider the presence of both ordinary or exotic matter, with special choices of w_1 and w_2 . Numerical integration of the orbits in this case, confirms the previous analysis, namely that the dominating value of w determines the final fate, capture or escape, of the particle trajectory. A more in depth analysis of the multi-component fluid scenario will make the object of future investigations.

4 Concluding Remarks

The motion of massive test particles interacting with a perfect fluid superposed to a Schwarzschild black hole is

investigated. The equation of state describing the fluid is taken as $p_0 = w\rho_0$, which corresponds to different physical scenarios for different values of w , from ordinary matter to exotic matter. The interaction with the fluid distribution is modeled by a force term entering the equations of motion given by the 4-momentum density observed in the particle's rest frame with a multiplicative constant factor expressing the strength of the interaction itself. Deviations from geodesic motion are analyzed for different values of the scattering parameter as well as of the parameter w . The latter turns out to be crucial in distinguishing among capture ($w + 1 > 0$) or escape ($w + 1 < 0$) from the black hole. The measurement of such effects may offer a new means of gleaning information on the nature of the cosmic matter surrounding astrophysical objects.

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